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A Rational Interpretation of the Dirac Equation for the Electron

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Abstract

Rationalization of the interpretation of the Dirac equation for the electron lies beyond the conventional scope of quantum mechanics. This difficulty motivates a revision of the system of quantum mechanics through which the indeterministic trait is eliminated from the system.

1. *Introduction*

Dirac's formulation of an equation for the electron in 1928 is often regarded as a superb example of synthetic work in quantum mechanics. Nevertheless, the concerned equation, which will be referred to as the Dirac equation from here on, seems to lie beyond the conventional scope of understanding of quantum mechanics. This observation is based on the following:

(a) The Dirac equation is formulated as covariant under the Lorentz transformation. However, as will be discussed in some detail in Appendix A, the concepts of velocity and position of a particle defined in the Lorentz transformation do not concur with the principle of uncertainty.

(b) As is shown in a paper published earlier, (Koga, 1975) and referred to as paper I in the present paper, the principle of superposition does not hold with respect to states satisfying the Dirac equation.

Since the principle of uncertainty and that of superposition are fundamental and essential in the conventional system of quantum mechanics, the Dirac equation cannot be comprehended rationally within the system. How is this difficulty resolved? Does the difficulty motivate a general revision of quantum mechanics by which the scope of quantum mechanics is broadened so that the Dirac equation is comprehended rationally? It is the purpose of the present paper to show that the revision is possible and that it enhances the significance of quantum mechanics.

In Section 2, the derivation of the relativistic and quantum-mechanical

Liouville equation from the Dirac equation is demonstrated. Utilizing the Liouville equation, in Section 3, a particular solution of the Dirac equation is shown to represent a wavelet of which the motion is determined as analogous to the motion of a classical-mechanical material point considered in the special-relativistic sense. The principle of uncertainty and that of superposition are not relevant with respect to the motion of the wavelet. The extent of feasibility of the conventional system of quantum mechanics is discussed.

It is of interest to note that the same revision of the conventional system of quantum mechanics is necessary also for rationalizing a treatment of a system consisting of many similar particles (Koga, 1972).

2. *The Liouville Equation Derived from the Dirac Equation*

In classical mechanics we often regard the Hamilton-Jacobi equation, the Liouville equation, and others as basic equations. It is obvious that those *partial* differential equations are not equivalent to Newton's equations of motion which are *ordinary* differential equations. Also, the Hamilton-Jacobi equation is not equivalent to the Liouville equation, and so on. It is only meant that if we choose a particular solution, e.g., of the Liouville equation, then the solution is interpretable as equivalent to a solution of Newton's equations of motion. The situation is well known (Whittaker, 1965). We assume that the situation is the same with respect to the Dirac equation: Only if we choose a particular solution of the Dirac equation obtained under certain restrictive conditions, may the solution represent the behaviour of the electron. The choice of the solution is guided not only by the correspondence principle but also by the following two conditions: (a) the interpretation of the state of an electron represented by the solution is compatible with the definition of the Lorentz transformation; (b) the interpretation is physically comprehensible without relying on the conventional theory of measurement which is based on the principle of indeterminacy and the principle of superposition. We note that the principle of superposition is not valid with respect to states satisfying the Dirac equation as is shown in paper I.

We may write for the Dirac equation

$$D_0\Psi = 0 \tag{2.1}$$

where

$$\Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4)$$

$$D_0 = \beta(i\hbar\,\partial/\partial t - eA_t) + \beta\boldsymbol{\alpha}\cdot(i\hbar c\,\partial/\partial \mathbf{r} + e\mathbf{A}) - mc^2 \tag{2.2}$$

and $(\mathbf{A}, iA_t)$ is a 4-vector potential. The other symbols are conventional as defined in Paper I. As noted in appendix B, $(\beta\boldsymbol{\alpha}, i\beta)$ are regarded as constituting a 4-vector.

As is demonstrated in Paper I (Section IV), the Dirac equation can be reduced to a set of four equations of which each contains only one of the four components of Ψ. The reduction is done only by increasing the order of those

differential equations and by ignoring time- and space-derivatives of (A, iA_t) of some higher orders. The complexity of the reduction increases as the complexity of (A, iA_t) as functions of time and space coordinates increases. Since the purpose of the present paper is to demonstrate a general approach taken along the line set forth in the beginning, we shall treat the simplest case where

$$(A, iA_t) = 0 \tag{2.3}$$

On operating

$$D_1 = \beta(i\hbar\,\partial/\partial t) + \beta\boldsymbol{\alpha}\cdot(i\hbar c\,\partial/\partial \mathbf{r}) + mc^2 \tag{2.4}$$

from the left hand side of equation (2.1), we have

$$D_1 D_0 \Psi = (\hbar^2\,\partial^2/\partial t^2 - \hbar^2 c^2\,\partial^2/\partial r^2 + m^2c^4)\Psi = 0 \tag{2.5}$$

Equation (2.5) is covariant under the Lorentz transformation. If we find a solution $\Psi^{(1)}$ of equation (2.5), then

$$\Psi^{(0)} = D_1 \Psi^{(1)} \tag{2.6}$$

is a solution of the Dirac equation, because

$$D_1 D_0 = D_0 D_1$$

We write for the Ψ's in equation (2.5)

$$\Psi_j^{(1)} = a_j \exp(iS_j/\hbar) \qquad (j = 1, 2, 3, 4) \tag{2.7}$$

where a_j and S_j are real functions of t and r. By substituting (2.7) in equation (2.5), we get the same equation for each of the Ψ's as for the others. Hence, subscript j may be omitted from here on. By separating the real and imaginary parts of the resultant equation, we get

$$\left(\frac{\partial S}{\partial(ct)}\right)^2 - (\operatorname{grad} S)^2 - \frac{\hbar^2}{a}\left(\frac{\partial^2 a}{\partial(ct)^2} - \Delta a\right) - m^2c^2 = 0 \tag{2.8}$$

$$\frac{\partial}{\partial(ct)}\left(a^2\,\frac{\partial S}{\partial(ct)}\right) - \operatorname{div}(a^2 \operatorname{grad} S) = 0 \tag{2.9}$$

We define Q by

$$Q = \hbar^2\,[\partial^2 a/\partial(ct)^2 - \Delta a]\,/a \tag{2.10}$$

for the sake of convenience. If a is a scalar, then Q is also a scalar under the Lorentz transformation. If S is a scalar also, equations (2.8) and (2.9) are covariant under the Lorentz transformation. If $\hbar = 0$ in equation (2.8), the equation may be regarded as a relativistic version of the Hamilton-Jacobi equation in classical mechanics. In order to make the interpretation of equation (2.8) as analogous to the ordinary interpretation of the Hamilton-Jacobi equation in mechanics, we must select a proper set of solutions of

equations (2.8) and (2.9). For such solutions of equations (2.8) and (2.9), we consider

$$S = S(\mathbf{r}, t, \mathbf{r}_0, t_0)$$
$$a = a(\mathbf{r}, t, \mathbf{r}_0, t_0) \tag{2.11}$$

in which $\mathbf{r}_0$ and t_0 are constants (footnote 1). We define

$$\mathbf{p} = \text{grad}\, S \tag{2.12}$$
$$E = -\partial S/\partial(ct) \tag{2.13}$$

We regard p as momentum, a 3-vector variable independent of $\mathbf{r}$, and E as energy independent of t. Obviously it holds that

$$d\mathbf{p}/d(ct) + \text{grad}\, E = 0 \tag{2.14}$$

according to (2.12) and (2.13). On substituting (2.12) and (2.13) in (2.8) and (2.9), we have for (2.8)

$$E^2 = p^2 + Q + m^2c^2 \tag{2.15}$$

and for (2.9)

$$[E(\partial/\partial(ct))_r + \mathbf{p} \cdot \text{grad}_t]a^2 = 0 \tag{2.16}$$

where subscripts r and t are attached in order to emphasize the independence between $\mathbf{r}$ and t. Equation (2.15) may be regarded as the equation of energy.

By means of (2.12) and (2.13), we may eliminate those constants $\mathbf{r}_0$ and t_0 from (2.11), obtaining

$$S = S(\mathbf{r}, t, \mathbf{p}, E)$$
$$a = a(\mathbf{r}, t, \mathbf{p}, E) \tag{2.17}$$

Then, noting that E is independent of t and $\mathbf{p}$ is independent of r, we have

$$\left(\frac{\partial a}{\partial(ct)}\right)_r = \left(\frac{\partial a}{\partial(ct)}\right)_{r,p} + \frac{d\mathbf{p}}{d(ct)} \cdot \frac{\partial a}{\partial \mathbf{p}}$$
$$(\text{grad}\, a)_t = (\text{grad}\, a)_{t,E} + (\text{grad}\, E)\, \partial a/\partial E \tag{2.18}$$

where the derivatives in the left-hand sides are those contained in (2.16), and the derivatives in the right-hand sides are made with respect to a given in (2.17). We substitute (2.18) in (2.16). In the result, the following manipulation is made with the help of (2.14) and (2.15):

$$E\frac{d\mathbf{p}}{d(ct)} \cdot \frac{\partial}{\partial \mathbf{p}} + \mathbf{p} \cdot \text{grad}\, E \frac{\partial}{\partial E} = -E\,(\text{grad}\, E) \cdot \frac{\partial}{\partial \mathbf{p}} - \mathbf{p} \cdot \frac{d\mathbf{p}}{d(ct)} \frac{\partial}{\partial E}$$
$$= -\frac{1}{2}(\text{grad}\, Q) \cdot \frac{\partial}{\partial \mathbf{p}} + \frac{1}{2}\frac{dQ}{d(ct)} \frac{\partial}{\partial E}$$

If S is the principal function satisfying the Hamilton–Jacobi equation in classical mechanics, $\mathbf{r}_0$ may be regarded as the value of $\mathbf{r}$ at the initial time t_0.

Finally we have for (2.16)

$$\left[E\frac{\partial}{\partial(ct)} + \mathbf{p}\cdot\operatorname{grad} + \frac{1}{2}\left(\frac{dQ}{d(ct)}\frac{\partial}{\partial E} - (\operatorname{grad} Q)\cdot\frac{\partial}{\partial \mathbf{p}}\right)\right]a^2 = 0 \qquad (2.19)$$

where the subscripts attached to derivatives have been removed.

When $(d\mathbf{r}, id(ct))$ is a 4-vector, $(\operatorname{grad}, -i\partial/\partial(ct))$ is transformed also as a 4-vector. Noting this, we see that equation (2.19) is covariant under the Lorentz transformation. A question remains with respect to Q. According to (2.10), Q is a scalar. If we introduce (2.18) in (2.10) and regard $(\mathbf{p}, iE)$ as a 4-vector, however, the qualification of Q to be a scalar is obscured, because the independence between $\mathbf{p}$ and $\mathbf{r}$ and also the independence between E and t are not necessarily preserved under the Lorentz transformation. This difficulty may be explained as follows: As is discussed in Appendix B, the Dirac equation is only conditionally covariant under the Lorentz transformation. This fact suggests that the Dirac equation contains some description of the internal structure of the electron for which the Lorentz transformation is not relevant, and the investigation made in the next section suggests that Q bears the very description.

Under conditions

$$m^2c^2 > (p^2 + Q), \qquad E > 0$$

and small quantities of higher orders being neglected, equations (2.15) and (2.19) yield respectively

$$cE = mc^2 + p^2/(2m) + Q/(2m) \qquad (2.20)$$

$$\left(\frac{\partial}{\partial t} + \frac{1}{2mc}\frac{dQ}{dt}\frac{\partial}{\partial E} - \frac{1}{2m}(\operatorname{grad} Q)\cdot\frac{\partial}{\partial \mathbf{p}} + \frac{\mathbf{p}}{m}\cdot\operatorname{grad}\right)a^2 = 0 \qquad (2.21)$$

If we eliminate mc^2 from the energy equation (2.20) and neglect $\partial^2 a/\partial(ct)^2$ in Q given by (2.10), and further assume that dQ/dt is insignificant, then equations (2.20) and (2.21) are derivable from Schrödinger's nonrelativistic wave equation (Koga, 1972).

3. *Particular Solutions of the Dirac Equation and the Liouville Equation*

In Section 2, the Liouville equation was derived from equation (2.5) where

$$D_0\Psi = 0$$

is the original Dirac equation. The relation between a solution of equation (2.5) and the corresponding solution of the Dirac equation is given by (2.6).

It is easily seen that equation (2.19) is satisfied by (footnote 2)

$$a = \frac{\exp(-\kappa|\mathbf{r}|)}{|\mathbf{r}|}\delta(\mathbf{p}) \qquad (3.1)$$

[2] Note that $\mathbf{p}\delta(\mathbf{p}) = 0$.

where

$$|\mathbf{r}| = (x^2 + y^2 + z^2)^{1/2}, \qquad \delta(\mathbf{p}) = \delta(p_x)\delta(p_y)\delta(p_z)$$

δ denoting Dirac's δ-function, and we note that, according to (2.10) and (2.15),

$$Q = -\hbar^2 \Delta a/a = -\hbar^2 \kappa^2 = \text{const.}$$
$$E^2 = m^2c^2 - \hbar^2 \kappa^2$$

By considering a coordinate system moving with velocity $\mathbf{u}$ relative to the original coordinate system, $\mathbf{r}$ and $\mathbf{p}$ in (3.1) are replaced respectively with

$$\mathbf{r}' = (\mathbf{r} - \mathbf{u}t)/(1 - u^2/c^2)^{1/2} \tag{3.2}$$

and

$$\mathbf{p}' = \mathbf{p} - \mathbf{u}E/c \tag{3.3}$$

Then we have

$$Q = [\hbar^2(\partial^2/\partial(ct)^2 - \Delta)a]/a$$
$$= -\hbar^2 \kappa^2$$

where κ is the same constant as given in (3.1). Accordingly

$$a = \frac{\exp(-\kappa |\mathbf{r}'|)}{|\mathbf{r}'|} \delta(\mathbf{p}') \tag{3.4}$$

is shown to satisfy equation (2.19).

It might be of interest to recall that for obtaining a wavelet by solving Schrödinger's nonrelativistic equation, it was necessary to define velocity $\mathbf{v}$ by $\mathbf{p} = m\mathbf{v}$ by invoking the principle of Galilean invariance (Koga, 1972). In the present case, it is necessary to define $\mathbf{u}$ by

$$\mathbf{p} = \mathbf{u}E/c$$

so that the priciple of Einsteinian invariance, i.e., of the special theory of relativity, is maintained. See Møller (1952, Chapter III).

By substituting the above in (2.15), we obtain

$$E^2 = m^2c^2 - \hbar^2 \kappa^2 + u^2E^2/c^2$$

or

$$E^2 = (m^2c^2 - \hbar^2 \kappa^2)/(1 - u^2/c^2) \tag{3.5}$$

If $\hbar^2 = 0$ in this relation, the relation is known with respect to a material point in the conventional special theory of relativity.

We now take for S

$$S = -Ect + \mathbf{p} \cdot \mathbf{r}$$

Then Ψ which satisfies equation (2.5), i.e., $\Psi^{(1)}$ in (2.7), is given by

$$\Psi_j^{(1)} = A_j a \exp(iS/\hbar + i\theta_j) \qquad (j = 1, 2, 3, 4) \tag{3.6}$$

where θ_j and A_j are real numbers to be given arbitrarily. But there is no Hilbert space in which $\Psi^{(1)}$ is a vector. The cause of the difficulty is twofold: first $\Psi^{(1)}$ has more than one component (Koga, 1975); secondly a contained in $\Psi^{(1)}$ has singularity as is seen in (3.4).

In order to obtain the solution of the Dirac equation, as corresponding to (3.6), we utilize relation (2.6) and obtain

$$\Psi^{(0)} = \left(\beta i\hbar \frac{\partial}{\partial t} + i\hbar c \, \beta \boldsymbol{\alpha} \cdot \frac{\partial}{\partial \mathbf{r}} + mc^2 \right) \Psi^{(1)} \tag{3.7}$$

where $\Psi^{(1)}$ is given by (3.6). The amplitude a of $\Psi^{(1)}$ is spherically symmetric with respect to $\mathbf{r}'$ in (3.4). But this condition does not hold for $\Psi^{(0)}$; the field represented by $\Psi^{(0)}$ is anisotropic. The mode of the anisotropy varies as the representation of spin matrices varies in the Dirac equation. The formation of a de Broglie wave in terms of $\Psi^{(1)}$ may be done in the same way as in the case of Schrödinger's equation given earlier (Koga, 1972).

6. *Concluding Remarks*

Since the principle of superposition is not valid with respect to solutions of the Dirac equation for the electron, the conventional interpretation of the equation made as based on the principle is not valid (Koga, 1975).

A rational interpretation of the concerned equation is obtained by considering the equation as corresponding to the Hamilton–Jacobi equation in classical mechanics. According to this investigation, our knowledge of the motion of the electron is deterministic; the principle of uncertainty and also the principle of indeterminacy are not relevant.

It is noted that this deterministic view of nature is also necessary for treating time-dependent processes taking place in a system consisting of many similar particles even in the nonrelativistic sense (Koga, 1972).

To sum up, the conventional formalism of quantum mechanics as based on the principle of uncertainty and that of superposition is to be feasible only for treating a system in particular cases in which the system may be represented by an ensemble in the non-relativistic sense.

Appendix A

Definition of the Lorentz Transformation

In the definition of the Lorentz transformation, the relative velocity between a pair of coordinate systems must be considered in the kinematical sense, so that the velocity may specify the kinematical relation between the two coordinate systems. In quantum mechanics, the velocity operator $\mathbf{v}$ of a particle is defined formally by

$$\mathbf{v} = \dot{\mathbf{r}} = (i/\hbar)(H\mathbf{r} - \mathbf{r}H) = \mathbf{p}/m$$

where $\mathbf{p}$ is the momentum operator and H the Hamiltonian (Dirac, 1967, section 69). The determination of velocity requires the determination of momentum. According to the principle of uncertainty, however, the determination of momentum is made only by forsaking the determination of position. On the other hand, the concept of velocity in the kinematical sense necessary for the Lorentz transformation requires the determination of position.

This situation is well known. See, for instance, Dirac (1967, p. 262). The situation gives rise to the following difficulty: in order to achieve the relativistic covariancy of a quantum-mechanical theory, we have to formulate the theory so as to embrace the definition of velocity which can be made only in the classical-mechanical and deterministic sense.

Appendix B

Covariancy of the Dirac Equation Under the Lorentz Transformation

In Paper I, Section II, we observed that, if the anisotropy of the electron structure is embodied in the Dirac equation, it should be represented by the anisotropy of the spin matrices. From this point of view, we must regard $(\beta\boldsymbol{\alpha}, i\beta)$ as a 4-vector and Ψ as a set of four scalars in equation (2.2), in order to make the Dirac equation covariant under the Lorentz transformation. By transforming formally $(\beta\boldsymbol{\alpha}, i\beta)$, we suppose that $((\beta\boldsymbol{\alpha})', i\beta')$ is obtained for the components in the new coordinate system. Then $\boldsymbol{\alpha}'$ is defined by

$$\boldsymbol{\alpha}' = \beta'(\beta\boldsymbol{\alpha})'$$

Thus $\boldsymbol{\alpha}'$ and β' are shown to satisfy the same relations as known between $\boldsymbol{\alpha}$ and β. For instance, $\alpha_x'^2 = \alpha_y'^2 = \alpha_z'^2 = 1$, etc. If we regard the Dirac equation as a set of four partial differential equations, however, those equations are not covariant under the concerned transformation. This situation is comprehensible: the anisotropy of the electron embodied in the Dirac equation should not rotate together with the coordinate system when the latter is rotated by us.

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